Mather problem and viscosity solutions in the stationary setting

Diogo A. Gomes and Elismar R. Oliveira

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Abstract

In this paper we discuss the Mather problem for stationary Lagrangians, that is Lagrangians $L : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}$, where Ω is a compact metric space on which $\mathbb{R}^n$ acts through an action which leaves L invariant. This setting allow us to generalize the standard Mather problem for quasi-periodic and almost-periodic Lagrangians. Our main result is the existence of stationary Mather measures invariant under the Euler-Lagrange flow which are supported in a graph. We also obtain several estimates for viscosity solutions of Hamilton-Jacobi equations for the discounted cost infinite horizon problem.

1 Introduction

Let M be a complete compact manifold, and $L : TM \to \mathbb{R}$ a C^3 Lagrangian, fiberwise strictly convex and coercive. A probability measure on TM is called holonomic if

$$\int_{TM} v \cdot D\varphi d\mu = 0,$$

for all $\varphi \in C^1(M)$. A central result in Aubry-Mather theory [Mn96] (see also [FS04]), is the fact that any holonomic probability measure μ on TM which minimizes the action $\int_{TM} L d\mu$ is supported on a Lipschitz graph and is invariant under the Euler-Lagrange flow. Certain results in Aubry-Mather theory have been extended for non-compact manifolds, see for instance [FM07] or [Mad06], but as far as the authors know, there is in the literature no satisfactory construction of Mather measures for general non-compact manifolds.

In this paper, rather than considering Lagrangians on the tangent bundle of compact manifolds, such as in the original paper of Mather [Mat91], we consider Lagrangians defined on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega$, where Ω is a suitable compact metric space on which $\mathbb{R}^n$ acts trough an action τ_x. The main result of this paper is Theorem 16, in which we establish the existence of stationary Mather measures invariant under the Euler-Lagrange flow.

Stationary ergodic problems were considered in [LS03] in the context of homogenization of random stationary ergodic Hamilton-Jacobi equations. The authors (in particular DG) are thankful to several enlightening discussions with P. Souganidis on this issue. Generalized Mather measures for stationary ergodic problems were also considered in the homogenization setting in [GV07]. The stationary ergodic setting was consider in [**?**] where the construction of critical (or critical approximate) viscosity solutions of Hamilton-Jacobi equations is carried out in detail for the one-dimensional case.

A simple example (taken from [LS03]) which illustrates the main difficulties in the stationary setting is the Lagrangian

$$L = \frac{|v|^2}{2} - \cos(x + \omega_1) - \cos(\sqrt{2}x + \omega_2).$$

Consider $\omega \in \mathbb{R}^2/\mathbb{Z}^2 \equiv \mathbb{T}^2$ as a fixed parameter. It would be natural, as in Mather's problem, to look for probability measures μ on $\mathbb{R}^n \times \mathbb{R}^n$ which minimize the action

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} L d\mu \tag{1}$$

under the holonomy constraint

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} v \cdot D_x \varphi d\mu = 0,$$

for all φ of class C^1, bounded with bounded derivatives. This problem can be solved explicitly, and in fact we have the following two cases: if there exists a solution $\bar{x}$ to the overdetermined system

$$\bar{x} + \omega_1 = 2\pi n, \qquad \sqrt{2}\bar{x} + \omega_2 = 2\pi n,$$

for some $n \in \mathbb{Z}$, the Mather measure on $\mathbb{R} \times \mathbb{R}$ is simply $\mu_0 = \delta_{\bar{x}}(x)\delta_0(v)$; otherwise there does not exist a Mather measure since $L > -1$ for all (x, v), and the infimum in (1) is easily shown to be -1.

To overcome these issues, which are due to the lack of compactness of $\mathbb{R}^n$, we will instead define stationary Mather measures as measures on $(v, \omega) \in \mathbb{R}^n \times \Omega$, which minimize the action and satisfy a suitable holonomy condition. It turns out that if Ω is compact and the Lagrangian satisfies certain stationarity hypothesis this is the natural way to generalize Mather measures. Before proceeding, we must make precise our framework.

Let Ω be a compact metric space, and let $L = L(x, v, \omega) : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a continuous Lagrangian, C^3 in the first two coordinates. The Lagrangian L is also required to be strictly convex and superlinear on the velocity v, and nonnegative. In our setting, this last condition can be achieved without changing of the nature the problem by adding a constant to L. We assume further that

$$L(x + y, v, \omega) - L(x, v, \omega) \leq |y| \left(C + CL(x, v, \omega)\right). \tag{2}$$

We suppose that there exists an action $\tau : \Omega \times \mathbb{R}^n \to \Omega$ which is continuous, satisfies the semigroup property

$$\tau_{x+y}\omega = \tau_x \tau_y \omega \text{ and } \tau_0(\cdot) = Id.$$

Since Ω is compact and the action is continuous, the action is uniformly transitive[1] in the following sense:

$$\forall \varepsilon > 0, \exists M > 0, \forall \omega_1, \omega_2 \in \Omega, \exists z \in \mathbb{R}^n, \text{ such that } |z| < M, \text{ and } d(\tau_z \omega_1, \omega_2) < \varepsilon.$$

A first example of such an action is the following: we take $\Omega = \mathbb{T}^d$, the d-dimensional torus, let $n < d$ and we will construct an action $\tau : \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{T}^d$. To start with, we identify the torus $\mathbb{T}^d$ with its universal covering $\mathbb{R}^d$, and consider a constant coefficient $d \times n$ matrix A. Assume that $\{Ax : x \in \mathbb{R}^n\}$ is dense in $\mathbb{T}^n$. Then we define

$$\tau_x \omega = \omega + Ax.$$

A second example is the following. We take Ω to be the space of all sequences $\omega = (\omega_k)$ on $\mathbb{T}^1$, endowed with the following metric:

$$d(\omega, \tilde{\omega}) = \sum_{k=1}^{\infty} 2^{-k} |\omega_k - \tilde{\omega}_k|.$$

[1] The authors are grateful to Albert Fathi that pointed out to us that uniform transitivity holds under the compactness assumption.

It is simple to verify that with this distance the space Ω is compact. A sequence λ of real numbers is called irrational if for any N the vector $(\lambda_1, \ldots, \lambda_N)$ is is linearly independent over the integers. Let $\bar{\lambda}$ be an irrational sequence. Define the following action from $\mathbb{R}$ into Ω by

$$\tau_x \omega = \omega + x\bar{\lambda}.$$

This action is also uniformly transitive.

A function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}$, is stationary if

$$\varphi(x + y, v, \omega) = \varphi(x, v, \tau_y(\omega)),\ \forall x, y \in \mathbb{R}^n,\ \omega \in \Omega.$$

We assume that the Lagrangian L is stationary.

Denote

$$\begin{aligned} C^1_s(\mathbb{R}^n \times \Omega) = \{\varphi : \mathbb{R}^n \times \Omega \to \mathbb{R},\ \text{stationary},\ C^1 \text{ in the first variable, continuous in} \\ \omega, \text{ and such that } D_x\varphi(0, \omega) \text{ is continuous in } \omega\}, \end{aligned}$$

with an analogous definition for $C^1_s(\mathbb{R}^n \times \mathbb{R}^n \times \Omega)$.

If the action is given as in the first example by $\tau_x\omega = \omega + Ax$, given $\psi : \mathbb{T}^d \to \mathbb{R}$, the function $\varphi(x, \omega) = \psi(\omega + Ax)$ is stationary, and, furthermore, $\varphi \in C^1_s$ if ψ is C^1. In the second example we can construct an example of a stationary function in the following way: let $\psi_k : \mathbb{T} \to \mathbb{R}$ be a sequence of periodic functions uniformly bounded in k. Let

$$\varphi(x, \omega) = \sum_k \psi_k(\omega_k + \bar{\lambda}_k x) 2^{-k} \frac{1}{1 + |\bar{\lambda}_k|}$$

Furthermore, if ψ_k is C^1 and its derivatives are uniformly bounded in k, $\varphi \in C^1_s$.

To motivate the stationary Mather problem, let $x(t)$ be a globally Lipschitz trajectory on $\mathbb{R}^n$. Let $\omega_0 \in \Omega$ is an arbitrary point. Consider ergodic averages to define an occupation measure μ on $\mathbb{R}^n \times \Omega$ corresponding to $x(\cdot)$ in the following way

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(x, \dot{x}, \omega_0) dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(0, \dot{x}, \tau_x\omega_0) dt \equiv \int_{\mathbb{R}^n \times \Omega} \phi(0, v, \omega) d\mu,$$

where the limit is taken trough an appropriate sequence. Of course, the measure μ could depend on the point ω_0 or the sequence through which the limit is taken. Nevertheless, such probabilities μ, satisfy an integral constraint, the holonomy condition:

$$\int_{\mathbb{R}^n \times \Omega} v \cdot D_x\varphi(0, \omega) d\mu = 0, \tag{3}$$

for any stationary function $\varphi \in C^1_s(\mathbb{R}^n \times \Omega)$.

The stationary Mather problem can be formulated as follows: minimize

$$\int_{\mathbb{R}^n \times \Omega} L(0, v, \omega) d\mu(v, \omega),$$

over all probability measures that satisfy the holonomy constraint (3). A minimizing measure for this problem is called a stationary Mather measure. A similar problem arises also in [GV07] for the homogenization of Hamilton-Jacobi equations.

Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a positive function such that

$$\lim_{|v|\to\infty} \frac{|v|}{\gamma(v)} = 0, \text{ and } \lim_{|v|\to\infty} \frac{L(0, v, \omega)}{\gamma(v)} = +\infty, \tag{4}$$

where the last limit is uniform in $\omega \in \Omega$ by compactness. We denote by $C^0_\gamma(\mathbb{R}^n \times \Omega)$ the set of the continuous functions ϕ with

$$\|\phi\|_\gamma = \sup_{\mathbb{R}^n \times \Omega} \frac{|\phi(v,\omega)|}{\gamma(v)} < \infty, \qquad \lim_{|v| \to \infty} \frac{|\phi(v,\omega)|}{\gamma(v)} \to 0.$$

We will need also to consider the discounted Mather problem, see [Gom08] for a discussion of related generalizations of Mather's problem. For that, let α be a positive number. Consider the operator $A : C^1_s(\mathbb{R}^n \times \Omega) \to C^0_\gamma(\mathbb{R}^n \times \Omega)$ given by

$$\varphi \to A^v\varphi(\omega) = v \cdot D_x\varphi(0,\omega) - \alpha\varphi(0,\omega).$$

The discounted stationary Mather problem consists in minimizing

$$\int_{\mathbb{R}^n \times \Omega} L(0,v,\omega) d\mu(v,\omega)$$

over all probability measures that satisfy the discounted holonomy constraint

$$\int_{\mathbb{R}^n \times \Omega} A^v\varphi(\omega) d\mu(v,\omega) = -\alpha \int_\Omega \varphi(0,\omega) d\nu(\omega), \tag{5}$$

for all $\varphi \in C^1_s(\mathbb{R}^n \times \Omega)$. A minimizing probability measure for this problem is called a discounted stationary Mather measure. The measure ν is called the trace of μ. If $\alpha = 0$ we call these measures stationary Mather measures.

The main result of this paper is the construction of stationary Mather measures invariant under the Euler-Lagrange flow. Usually, this flow is defined in $\mathbb{R}^n \times \mathbb{R}^n$. However, since the stationary Mather measures are measures on $\mathbb{R}^n \times \Omega$ we must now discuss the natural extension of the Euler-Lagrange flow to this space.

Given a stationary vector field $W : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n \times \mathbb{R}^n$, let $\Phi = (\Phi_1, \Phi_2) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ be its flow. We define the flow $\Psi : \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n \times \Omega$ induced by W in $\mathbb{R}^n \times \Omega$ as

$$\Psi(t,v,\omega) = (\Phi_2(t,0,v,\omega), \tau_{\Phi_1(t,0,v,\omega)}\omega).$$

We denote by $C^1_b(\mathbb{R}^n \times \Omega)$ the set of bounded continuous functions $\phi(v,\omega)$ in $\mathbb{R}^n \times \Omega$ such that $D_v\phi(v,\omega)$ is also continuous and bounded. A measure μ is invariant under the flow Ψ if,

$$\int_{\mathbb{R}^n \times \Omega} \phi(\Psi(t,v,\omega)) d\mu(v,\omega) = \int_{\mathbb{R}^n \times \Omega} \phi(v,\omega) d\mu(v,\omega),$$

for all $\phi \in C^1_b(\mathbb{R}^n \times \Omega)$ and for all $t \in \mathbb{R}$.

Let μ be a measure in $\mathbb{R}^n \times \Omega$ and $W : \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ be a stationary vector field in $\mathbb{R}^n \times \mathbb{R}^n$. Then μ is invariant under the flow induced by W in $\mathbb{R}^n \times \Omega$, if and only if,

$$\int_{\mathbb{R}^n \times \Omega} \nabla\hat{\phi}(0,v,\omega) \cdot W(0,v,\omega) d\mu(v,\omega) = 0, \tag{6}$$

(where the gradient in the previous formula is taken both in x and v) for all $\hat{\phi} \in \hat{C}^0_\gamma(\mathbb{R}^n \times \mathbb{R}^n \times \Omega)$. A proof for this classical fact for the case of vector fields on a manifold M can be found, for instance, in [BG08]. The proof in our setting follows exactly along the same lines and we will omit it.

In this paper we will need to consider the discounted Lagrangian $L_\alpha \equiv e^{-\alpha t} L(x,v)$. The corresponding Euler-Lagrange equation is

$$\frac{d}{dt} D_v L(x,v,\omega) = D_x L(x,v,\omega) + \alpha D_v L, \tag{7}$$

for each $\omega \in \Omega$. For $\alpha = 0$ we obtain the usual Euler-Lagrange equations. We have a ω-parametric Lagrangian vector field W^{L_α}, that is given by:

$$W^{L_\alpha} = \begin{cases} X^{L_\alpha}(x, v, \omega) = v \\ Y^{L_\alpha}(x, v, \omega) = (D^2_{vv}L)^{-1}(D_x L + \alpha D_v L - D_{xv}Lv). \end{cases}$$

We say that a measure μ in $\mathbb{R}^n \times \Omega$ is invariant under the Euler-Lagrange flow if it is invariant under the flow Ψ^α induced by W^{L_α} in $\mathbb{R}^n \times \Omega$.

The outline of this paper is as follows: in section 2 we describe briefly the duality theory for the stationary Mather problem and its connections with viscosity solutions of Hamilton-Jacobi equations. The proofs of some the results, since they are standard, are outlined for completeness in appendix A. In section 3 we make some formal computations in the spirit of [EG01]. These computations suggest that for certain discounted stationary Mather measures one may be able to extend the regularity results in [EG01]. Holonomic discounted stationary Mather measures are constructed in section 4. Using these measures we obtain regularity results for viscosity solutions in section 5. These imply that the discounted stationary Mather measures are supported in a (partially) Lipschitz graph whose Lipschitz constant is independent of the discount factor α. Finally in the last section we construct stationary Mather measures invariant under the Euler-Lagrange flow.

2 Duality and viscosity solutions

The stationary Mather problem is an infinite dimensional linear programming problem. As usual in these problems (see [Gom08], for instance), the duality theory plays an important role and will be developed in this section.

Theorem 1. Let ν be a probability measure on Ω and $\alpha \geq 0$. Define

$$\overline{H}_\alpha = \inf \int_{\mathbb{R}^n \times \Omega} L(0, v, \omega) d\mu(v, \omega), \tag{8}$$

where the infimum is taken over all probability measures on $\mathbb{R}^n \times \Omega$ which satisfy the discounted holonomy condition (5). Let

$$\mathcal{H}(\varphi, x, \omega) = \sup_{v \in \mathbb{R}^n} (-A^v(\varphi)(x, \omega) - L(x, v, \omega)) = H(x, D_x\varphi(x, \omega), \omega) + \alpha\varphi(x, \omega),$$

with $H(x, p, \omega) = \sup_{v \in \mathbb{R}^n}(-p \cdot v - L(x, v, \omega))$.

Then, the infimum in (8) is achieved at some probability measure μ satisfying (5) and furthermore

$$\overline{H}_\alpha = -\inf_{\varphi \in C^1_s} \sup_{\Omega} \left\{ -\alpha \int_\Omega \varphi d\nu + \mathcal{H}(\varphi, 0, \omega) \right\}. \tag{9}$$

The proof of this Theorem is similar to analogous results in [Gom08], for instance. For completeness, however, we present the proof in the Appendix A.

In this paper we will need to consider viscosity solutions to the equation

$$\mathcal{H}^\alpha(u, 0, \omega) \equiv H(0, D_x u_\alpha(0, \omega), \omega) + \alpha u_\alpha(0, \omega) = 0. \tag{10}$$

As in the standard Mather problem, viscosity solutions yield important information concerning the value of the variational problem (8), and help characterize the support of the measure.

Before we proceed, we make some remarks concerning the regularization by convolution of stationary functions.

Remark 1. To approximate a stationary function $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$ by smooth stationary functions we are going to use a convolution with a standard mollifier $\eta^\varepsilon : \mathbb{R}^n \to \mathbb{R}$, that is, η compactly supported, $\eta^\varepsilon(x) = \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$, and $\int_{\mathbb{R}^n} \eta(x)dx = 1$. We define the convolution between u and η^ε by

$$u^\varepsilon(x,\omega) = \int_{\mathbb{R}^n} u(x, \tau_y\omega)\eta^\varepsilon(y)dy.$$

Observe that, $u^\varepsilon \in C^1_s$. Moreover, we have

$$\frac{\partial u^\varepsilon}{\partial x}(x,\omega) \cdot v = -\int_{\mathbb{R}^n} u(x, \tau_y\omega) D_y\eta^\varepsilon(y) \cdot v \, dy.$$

We consider two different types of viscosity solutions for $\mathcal{H}(u, 0, \omega) = \lambda$. Firstly recall the usual definition of viscosity solution: a function $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$, continuous in x (not necessarily C^1) for each $\omega \in \Omega$, is a *viscosity solution in* x of $\mathcal{H}(u, x, \omega) = \lambda$ if for each $\omega_0 \in \Omega$, any C^1 function $\psi : \mathbb{R}^n \to \mathbb{R}$ and any $x_0 \in \mathbb{R}^n$ such that $u(x, \omega_0) - \psi(x)$ has a strict local minimum (resp. maximum) at x_0 with $u(x_0, \omega_0) - \psi(x_0) = 0$ we have

$$\mathcal{H}(\psi, x_0, \omega_0) \geq \lambda \text{ (resp. } \leq \lambda).$$

For our purposes we need a modified version of viscosity solution: a stationary (not necessarily C^1) function $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$, continuous in Ω, is a *viscosity solution in* ω of $\mathcal{H}(u, 0, \omega) = \lambda$ if for any $\varphi \in C^1_s(\mathbb{R}^n \times \Omega)$ and any point $\omega_0 \in \Omega$ such that $u(0, \omega) - \varphi(0, \omega)$ has a local minimum (resp. maximum) at ω_0 with $u(0, \omega_0) - \varphi(0, \omega_0) = 0$ we have

$$\mathcal{H}(\varphi, 0, \omega_0) \geq \lambda \text{ (resp. } \leq \lambda).$$

Proposition 2. Suppose that $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is a viscosity solution in x of $\mathcal{H}(u, 0, \omega) = \lambda$ and assume furthermore that u is stationary and continuous in Ω. Then u is also a viscosity solution in ω of $\mathcal{H}(u, 0, \omega) = \lambda$.

Proof. Let $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a viscosity solution in x of $\mathcal{H}(u, 0, \omega) = \lambda$. Consider an arbitrary function $\varphi \in C^1_s(\mathbb{R}^n \times \Omega)$ and a point $\omega_0 \in \Omega$ such that $u(0, \omega) - \varphi(0, \omega)$ has a local minimum (resp. maximum) and $u(0, \omega_0) - \varphi(0, \omega_0) = 0$. Define $\psi(x) = \varphi(x, \omega_0)$. We claim that $u(x, \omega_0) - \psi(x)$ has a local minimum (resp. maximum) in $x_0 = 0 \in \mathbb{R}^n$. In fact,

$$\begin{aligned} u(x, \omega_0) - \psi(x) &= u(x, \omega_0) - \varphi(x, \omega_0) = u(0, \tau_x\omega_0) - \varphi(0, \tau_x\omega_0) \\ &\geq u(0, \omega_0) - \varphi(0, \omega_0) = u(0, \omega_0) - \psi(0), \quad \text{(resp. } \leq .) \end{aligned}$$

Then, because u is a viscosity solution in x we have

$$\mathcal{H}(\psi, 0, \omega_0) = \mathcal{H}(\varphi, 0, \omega_0) \geq \lambda \quad \text{(resp. } \leq \lambda).$$

□

Consider the infinite horizon optimal control problem

$$u_\alpha(x, \omega) = \inf_{x(0)=x} \int_0^{+\infty} e^{-\alpha t} L(x(t), \dot{x}(t), \omega)dt, \tag{11}$$

where the infimum is taken over all globally Lipschitz trajectories with initial condition $x(0) = x$. Then $u_\alpha : \mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfies the dynamic programing principle

$$u_\alpha(x, \omega) = \inf_{x(0)=x} \left(\int_0^T e^{-\alpha t} L(x(t), \dot{x}(t), \omega)dt + e^{-\alpha T} u_\alpha(x(T), \omega) \right), \tag{12}$$

among all globally Lipschitz trajectories with initial condition $x(0) = x$. It is standard, see [BCD97], that the function u_α is a viscosity solution of $\mathcal{H}(\varphi, 0, \omega) = 0$ in x. Furthermore, the optimal trajectories are solutions to the discounted Euler-Lagrange equations (7). Finally, for $0 < t < T$ we have additionally that $D_x u_\alpha(x(t))$ exists and

$$\dot{x}(t) = -D_p H(D_x u_\alpha(x(t)), x(t)).$$

The next proposition is also a well known result, see, for instance, [BCD97] for similar results:

Proposition 3. For each ω fixed, let $u_\alpha(x, \omega)$ be a viscosity solution (in x) of

$$\mathcal{H}^\alpha(u, x, \omega) = H(x, D_x u_\alpha(x, \omega), \omega) + \alpha u_\alpha(x, \omega) = 0. \tag{13}$$

Then αu_α is uniformly bounded and u_α is uniformly Lipschitz in x, as $\alpha \to 0$.

Using standard techniques we can establish the following proposition, whose proof is presented in appendix B:

Proposition 4. Let $u_\alpha : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a solution of (13). Then u_α is a viscosity solution (in ω) of $\mathcal{H}(\varphi, 0, \omega) = 0$, and $u_\alpha(0, \omega)$ is Lipschitz in ω with Lipschitz constant (in ω) bounded by K/α, where K is independent of α, for all $\alpha \geq 0$.

Proposition 5. Let u_α be a viscosity solution in ω of (13) Then

$$\inf_{\varphi \in C^1_s} \sup_{\omega \in \Omega} \left\{ -\alpha \int_\Omega \varphi(0, \omega) d\nu(\omega) + \mathcal{H}^\alpha(\varphi, 0, \omega) \right\} = -\alpha \int_\Omega u_\alpha(0, \omega) d\nu(\omega).$$

Proof. Consider a viscosity solution u_α of (13). Then for any $\varphi \in C^1_s$ there exists a point ω_φ of minimum for $u_\alpha(0, \omega) - \varphi(0, \omega)$. Consider $\varphi'(x, \omega) = \varphi(x, \omega) + (u_\alpha - \varphi)(0, \omega_\varphi)$. Then $u_\alpha(0, \omega) - \varphi'(0, \omega)$ has a minimum equal to 0 in ω_φ.

Since u_α is a viscosity solution we have $\mathcal{H}^\alpha(\varphi', 0, \omega_\varphi) \geq 0$ or equivalently

$$\mathcal{H}^\alpha(\varphi, 0, \omega_\varphi) + \alpha(u_\alpha - \varphi)(0, \omega_\varphi) \geq 0.$$

Therefore

$$-\alpha \int_\Omega \varphi(0, \omega) d\nu + \mathcal{H}^\alpha(\varphi, 0, \omega_\varphi) + \alpha(u_\alpha - \varphi)(0, \omega_\varphi) \geq -\alpha \int_\Omega \varphi(0, \omega) d\nu,$$

which implies

$$\sup_{\omega \in \Omega} -\alpha \int_\Omega \varphi(0, \omega) d\nu + \mathcal{H}^\alpha(\varphi, 0, \omega) \geq -\alpha \int_\Omega \varphi(0, \omega) d\nu + \alpha(u_\alpha - \varphi)(0, \omega_\varphi),$$

and so

$$\sup_{\omega \in \Omega} -\alpha \int_\Omega \varphi(0, \omega) d\nu + \mathcal{H}^\alpha(\varphi, 0, \omega) \geq -\alpha \int_\Omega u_\alpha(0, \omega) d\nu,$$

which finally yields

$$\inf_{\varphi \in C^1_s} \sup_{\omega \in \Omega} \left\{ -\alpha \int_\Omega \varphi(0, \omega) d\nu(\omega) + \mathcal{H}^\alpha(\varphi, 0, \omega) \right\} \geq -\alpha \int_\Omega u_\alpha(0, \omega) d\nu.$$

In order to get the other inequality we use the functions $u^\varepsilon = u_\alpha * \eta_\varepsilon$. Then $\mathcal{H}^\alpha(u^\varepsilon, 0, \omega) \leq o(1)$ owing to the convexity of the Hamiltonian and the uniform Lipschitz estimates on u_α, we have

$$\inf_{\varphi \in C^1_s} \sup_{\omega \in \Omega} \left\{ -\alpha \int_\Omega \varphi(0, \omega) d\nu(\omega) + \mathcal{H}^\alpha(\varphi, 0, \omega) \right\} \leq o(1) - \alpha \int_\Omega u^\varepsilon(0, \omega) d\nu.$$

Then, the inequality desired is obtained by sending ε to 0, and ends the proof. $\square$

Corollary 6. We have

$$\bar{H}_\alpha = \alpha \int_\Omega u_\alpha(0,\omega) d\nu$$

where u_α is the unique viscosity solution of $H(0, D_x u_\alpha(0,\omega), \omega) + \alpha u_\alpha(0,\omega) = 0$.

Proof. In fact, if we apply Proposition 5 we have the formula

$$\inf_{\varphi \in C^1_s} \sup_{\omega \in \Omega} \left\{ -\alpha \int_\Omega \varphi(0,\omega) d\nu(\omega) + \mathcal{H}^\alpha(\varphi, 0, \omega) \right\} = -\alpha \int_\Omega u_\alpha(0,\omega) d\nu(\omega).$$

Remembering that $\bar{H}_\alpha = -\inf_{\varphi \in C^1_s} \sup_{\omega \in \Omega} -\alpha \int_\Omega \varphi(0,\omega) d\nu(\omega) + \mathcal{H}^\alpha(\varphi, 0, \omega)$, we get

$$\bar{H}_\alpha = \alpha \int_\Omega u_\alpha(0,\omega) d\nu.$$

□

We state next, without proof, a partial converse to Proposition 2. The proof is rather technical and, in this paper, its only a application is in Remark 2.

Proposition 7. Suppose that,

(a) There exists $\delta > 0$ such that, for all $x \neq 0$ with $|x| < \delta$, $\tau_x(\cdot) : \Omega \to \Omega$ does not have fixed points.

(b) For each $\omega_0 \in \Omega$, there exists $\delta > 0$ and a set $\Sigma_\delta(\omega_0) \ni \omega_0$, such that, for all $x \neq 0$ with $|x| < \delta$, and $\omega_1, \omega_2 \in \Sigma_\delta(\omega_0)$, if $\tau_x(\omega_1) = \omega_2$ then $\omega_1 = \omega_2$.

(c) The set

$$\mathcal{U}_\delta(\omega_0) = \{\tau_x(\omega) | \, \omega \in \Sigma_\delta(\omega_0), \ |x| < \delta/2\} \tag{14}$$

is an open neighborhood of ω_0.

If $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is a viscosity solution in ω of $\mathcal{H}(u, 0, \omega) = \lambda$ then u is also a viscosity solution in x of $\mathcal{H}^\alpha(u, 0, \omega) = \lambda$.

Remark 2. Note that in some cases $\mathcal{H}^\alpha(u, \omega) = \lambda$ does not admit viscosity solutions in ω, as pointed out in [LS03]. In their example $\Omega = \mathbb{T}^2$, $L = L(x, v, \omega) : \mathbb{R} \times \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$ is the Lagrangian given by $L(x, v, \omega) = \frac{1}{2}v^2 + \cos(\omega_1 + x) + \cos(\omega_2 + \sqrt{2}x)$, with the associated Hamiltonian $H(x, p, \omega) = \frac{1}{2}p^2 - \cos(\omega_1 + x) - \cos(\omega_2 + \sqrt{2}x)$, and the action $\tau : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2$ is given by $\tau_x(\omega_1, \omega_2) = (\omega_1 + x, \omega_2 + \sqrt{2}x)$.

In this case the viscosity solutions in x are unbounded. So, if there where a viscosity solution in ω, then it would be a solution in x by Proposition 7. By compactness, any stationary continuous function is bounded, which would be a contradiction.

3 Some formal computations

In this section we adapt the formal computations in [EG01] to motivate the regularity results in the following sections. Consider the periodic case of a C^2 Lagrangian $L : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, given by $L(x, v) = \frac{1}{2}v^2 - V(x)$, and the associated Hamiltonian $H(x, p) = \frac{1}{2}p^2 + V(x)$. The stationary case follows along the same lines, as we will see in later sections.

Let u be a solution to the discounted Hamilton-Jacobi equation $\frac{1}{2}u_x^2 + V(x) + \alpha u = 0$. Let μ_α be a discounted Mather measure with trace θ_α and such that the projection of μ_α in the x coordinated is denoted by θ, that is,

$$\int_{\mathbb{T} \times \mathbb{R}} \varphi(x) d\mu_\alpha = \int_{\mathbb{T}} \varphi(x) d\theta.$$

Note that θ in general does not agree with θ_α. In this section we assume that μ_α has the special property that $\theta_\alpha = \theta$. Under this assumption μ_α is holonomic, that is

$$\int_{\mathbb{T}\times\mathbb{R}} v\varphi_x(x)d\mu_\alpha = 0,$$

for all C^1 periodic function $\varphi(x)$.

We will first show that μ_α almost every $(x,v) \in \mathbb{T}\times\mathbb{R}$, we have $v = -u_x(x)$. To see this we will argue by contradiction. In this case if $v \neq -u_x(x)$, there would exist a set of positive measure μ_α in which

$$L(x,v) + vu_x > -H(u_x, x).$$

Since $L(x,v) + vu_x \geq -H(u_x,x)$, integrating with respect to μ_α yields

$$\int_{\mathbb{T}\times\mathbb{R}} Ld\mu_\alpha + \int_{\mathbb{T}\times\mathbb{R}} vu_x d\mu_\alpha > \alpha\int_{\mathbb{T}\times\mathbb{R}} ud\mu_\alpha.$$

This would yield

$$\int_{\mathbb{T}\times\mathbb{R}} Ld\mu_\alpha > \alpha\int_{\mathbb{T}} ud\theta_\alpha,$$

which contradicts the optimality condition.

Therefore the holonomy constraint can be written as

$$\int_{\mathbb{R}} (u_x\varphi_x + \alpha\varphi)d\theta(x) = \alpha\int_{\mathbb{R}} \varphi d\theta_\alpha(x).$$

By differentiating twice the Hamilton-Jacobi equation we have $u_x(u_{xx})_x + u_{xx}^2 + V''(x) + \alpha u_{xx} = 0$. Integrating with respect to μ_α yields

$$\int_{\mathbb{R}} (u_x(u_{xx})_x + u_{xx}^2 + V''(x) + \alpha u_{xx})d\mu_\alpha = 0,$$

or, equivalently,

$$\int_{\mathbb{R}} u_x(u_{xx})_x d\theta(x) + \int_{\mathbb{R}} u_{xx}^2 + \alpha u_{xx}d\theta(x) = -\int_{\mathbb{R}} V''(x)d\theta(x).$$

Since the trace of μ_α, θ_α is equal to its projection $\theta(x)$, then the measure μ_α is holonomic and so

$$\int_{\mathbb{R}} u_x(u_{xx})_x d\theta(x) = 0.$$

Using $-\alpha u_{xx} \leq \frac{1}{2}u_{xx}^2 + \frac{1}{2}\alpha^2$ we get,

$$\begin{aligned}\int_{\mathbb{R}} u_{xx}^2 d\theta(x) &= -\int_{\mathbb{R}} V''(x)d\theta(x) - \int_{\mathbb{R}} \alpha u_{xx}d\theta(x)\\ &\leq -\int_{\mathbb{R}} V''(x)d\theta(x) + \int_{\mathbb{R}} \frac{1}{2}u_{xx}^2 + \frac{1}{2}\alpha^2 d\theta(x),\end{aligned}$$

which yields a $L^2(\theta)$ bound for u_{xx}:

$$\int_{\mathbb{R}} u_{xx}^2 d\theta(x) \leq \int_{\mathbb{R}} \alpha^2 - 2V''(x)d\theta(x).$$

In order to derive L^∞ estimates to u_{xx} we proceed as follows: first we multiply the second derivative of the Hamilton-Jacobi equation by a function $\Psi'(u_{xx})$,

$$\int_{\mathbb{R}} u_x(u_{xx})_x\Psi'(u_{xx})d\theta(x) + \int_{\mathbb{R}} \left[u_{xx}^2 + \alpha u_{xx} + V''(x)\right]\Psi'(u_{xx})d\theta(x) = 0.$$

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be such that

$$\Psi'(x) = \begin{cases} 1 & \text{if } x \leq -\lambda \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is fixed. Choose $\Phi(x) = \Psi'(x)$ (actually one should to use a C^∞ approximation of $\Psi'(x)$). Observe that $(u_{xx})_x \Phi(u_{xx}) = \Psi(u_{xx})_x$ and so $\int_{\mathbb{R}} u_x (u_{xx})_x \Psi'(u_{xx}) d\theta(x) = 0$. Define $A = \{x | u_{xx} \leq -\lambda\}$. Thus,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \left(u_{xx}^2 + V''(x) + \alpha u_{xx}\right) \Phi(u_{xx}) d\theta(x) \\ &= \int_A \left(u_{xx}^2 + V''(x) + \alpha u_{xx}\right) d\theta(x). \end{aligned}$$

Since $u_{xx} \leq -\lambda$, and using $\alpha u_{xx} \leq -\frac{1}{2} u_{xx}^2 - \frac{1}{2}\alpha^2$, one can show that, $0 \geq (\frac{\lambda^2}{2} - \frac{1}{2}\alpha^2 + c)\theta(A)$, where, $|V''| \leq c$. Since λ is arbitrary , we get $\theta(A) = 0$. Thus, there exists $\lambda > 0$, such that, $u_{xx} > -\lambda$, θ-a.e.

The solutions of $\alpha u + \frac{1}{2} u_x^2 + V(x) = 0$ are semi-concave (this is a standard result, see [BCD97] or the survey paper [BG08]), so we get that there exists $\beta > 0$ such that $u_{xx} < \beta$, and so, for some $C > 0$, $|u_{xx}| < C$, θ almost everywhere.

4 Holonomic discounted stationary Mather measures

Motivated by the formal computations in the previous section, we will now establish the existence of holonomic discounted stationary Mather measures. In the paper [FCG08], these measures were called invariant, we did not keep this name here to avoid confusion with invariance with respect to Euler- Lagrange equations.

Given a probability measure ν, and a corresponding discounted stationary Mather measure μ with trace ν, we say that μ is a holonomic discounted stationary Mather measure if

$$\int_{\mathbb{R}^n \times \Omega} \varphi(0, \omega) d\mu(v, \omega) = \int_\Omega \varphi(0, \omega) d\nu(\omega),$$

for all $\varphi \in C_s^1(\mathbb{R}^n \times \Omega)$. In particular, μ satisfies the undiscounted holonomy constraint.

Theorem 8. There exists a holonomic discounted stationary Mather measure.

Proof. Fix $\omega \in \Omega$. Consider a sequence $T_n \to \infty$ and a sequence $x_n(t)$ of minimizing trajectories for the dynamic programing principle (12), that is,

$$u_\alpha(0, \omega) = \int_0^{T_n} e^{-\alpha t} L(x_n(t), \dot{x}_n(t), \omega) dt + e^{-\alpha T_n} u_\alpha(x_n(T_n), \omega).$$

Because u_α is Lipschitz and

$$\dot{x}_n = -D_p H(D_x u^\alpha(x_n(t)), x_n(t))$$

the $|\dot{x}_n|$ is uniformly bounded.

Define a probability measure μ by

$$\int_{\mathbb{R}^n \times \Omega} \phi(v, \omega) d\mu(v, \omega) = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \phi(\dot{x}_n, \tau_{x_n(t)} \omega) dt,$$

for any $\phi \in C_\gamma^0(\mathbb{R}^n \times \Omega)$, where the limit is taken through an appropriate subsequence. This sublimit exists and is a probability measure because Ω is compact and $|\dot{x}_n|$ is uniformly bounded.

Let $\varphi \in C^1_s$. Observe that $\frac{d}{dt}\varphi(x_n(t),\omega) = \dot{x}_n(t)\cdot D_x\varphi(0,\tau_{x_n(t)}\omega)$. So, if $\phi(v,\omega) = v\cdot D_x\varphi(0,\omega)$, then

$$\int_{\mathbb{R}^n\times\Omega} \phi(v,\omega)d\mu(v,\omega) = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} \dot{x}_n(t)\cdot D_x\varphi(0,\tau_{x_n(t)}\omega)dt$$
$$= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\frac{d}{dt}\varphi(x_n(t),\omega)dt = \lim_{n\to\infty}\frac{\varphi(x_n(T_n)) - \varphi(x_n(0))}{T_n} = 0.$$

Since $A^v\varphi = v\cdot D_x\varphi(0,\omega) - \alpha\varphi(0,\omega)$,

$$\int_{\mathbb{R}^n\times\Omega} A^v\varphi d\mu = \int_{\mathbb{R}^n\times\Omega} v\cdot D_x\varphi(0,\omega) - \alpha\varphi(0,\omega)d\mu(v,\omega)$$
$$= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\dot{x}_n(t)\cdot D_x\varphi(0,\tau_{x_n(t)}\omega) - \alpha\varphi(0,\tau_{x_n(t)}\omega)dt$$
$$= -\alpha\lim_{T_n\to\infty}\frac{1}{T_n}\int_0^{T_n}\varphi(0,\tau_{x_n(t)}\omega)dt = -\alpha\int_\Omega\varphi(0,\omega)d\nu(\omega),$$

where ν is given by,

$$\int_\Omega g(\omega)d\nu(\omega) = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} g(\tau_{x_n(t)}\omega)dt,$$

for all continuous function $g:\Omega\to\mathbb{R}$. In particular, $\int_{\mathbb{R}^n\times\Omega}\varphi(0,\omega)d\mu(v,\omega) = \int_\Omega\varphi(0,\omega)d\nu(\omega)$.

We must to prove that μ is minimizing. To do so, fix first n and consider a partition $\{0 = t_0, t_1, ..., t_{N-1} = T_n\}$ of $[0,T_n]$, where $t_{i+1} = t_i + h$, and $h = T_n/N$. The restriction of $x_n(t)$ to each sub-interval is minimizing, i.e.,

$$u_\alpha(x_n(t_i),\omega) = \int_{t_i}^{t_{i+1}} e^{-\alpha(t-t_i)}L(x_n(t),\dot{x}_n(t),\omega)dt + e^{-\alpha h}u_\alpha(x_n(t_{i+1}),\omega).$$

We have,

$$\sum_{i=0}^{i=N-1} u_\alpha(x_n(t_i),\omega) - e^{-\alpha h}u_\alpha(x_n(t_{i+1}),\omega) =$$
$$= \sum_{i=0}^{i=N-1} u_\alpha(x_n(t_i),\omega) - u_\alpha(x_n(t_{i+1}),\omega) + (1-e^{-\alpha h})u_\alpha(x_n(t_{i+1}),\omega) =$$
$$= u_\alpha(x_n(0),\omega) - u_\alpha(x_n(T),\omega) + \alpha(\frac{1-e^{-\alpha h}}{\alpha h})\sum_{i=0}^{i=N-1} hu_\alpha(x_n(t_{i+1}),\omega).$$

Sending $h\to 0$ we get

$$\lim_{h\to 0}\sum_{i=0}^{i=N-1} u_\alpha(x_n(t_i),\omega) - e^{-\alpha h}u_\alpha(x_n(t_{i+1}),\omega) = u_\alpha(x_n(0),\omega) - u_\alpha(x_n(T_n),\omega) + \alpha\int_0^{T_n} u_\alpha(0,\tau_{x_n(t)}\omega)dt.$$

On the other hand, we have

$$\lim_{h\to 0}\sum_{i=0}^{i=N-1}\int_{t_i}^{t_{i+1}} e^{-\alpha(t-t_i)}L(x_n(t),\dot{x}_n(t),\omega)dt = \int_0^{T_n} L(x_n(t),\dot{x}_n(t),\omega)dt.$$

Thus,

$$\alpha \int u_\alpha(0,\omega)d\nu =$$
$$\lim_{n\to\infty} \frac{1}{T_n}\left\{u_\alpha(x_n(0),\omega) - u_\alpha(x_n(T_n),\omega) + \alpha\int_0^{T_n} u_\alpha(0,\tau_{x_n(t)}\omega)dt\right\}$$
$$= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} L(x_n(t),\dot{x}_n(t),\omega)dt = \int_{\mathbb{R}^n\times\Omega} L(0,v,\omega)d\mu(v,\omega).$$

By Corollary 6 we have $\bar{H}_\alpha = \alpha\int u_\alpha(0,\omega)d\nu$. Thus μ is minimizing. □

We should note here that the Theorem does not assert uniqueness. Furthermore the measure μ may depend on the choice of $\omega \in \Omega$ or in the particular sequence we choose to extract the weak limits. For our purposes, however, existence is sufficient.

Theorem 9. Let μ_α be a holonomic discounted Mather measure as constructed in theorem 8. Then μ_α is invariant under the discounted Euler-Lagrange flow.

Proof. It suffices to prove that for any bounded function $\phi(x,v,\omega) \in C_s^1(\mathbb{R}^n\times\mathbb{R}^n\times\omega)$ we have

$$\int_{\mathbb{R}^n\times\Omega} W^{L_\alpha}\nabla_{x,v}\phi(0,v,\omega)d\mu_\alpha = 0.$$

This follows, from the identity

$$\begin{aligned}\phi(x_n(T_n),\dot{x}_n(T_n),\omega) - \phi(x_n(0),\dot{x}_n(0),\omega) &= \int_0^{T_n}\frac{d}{dt}\phi(x_n(t),\dot{z}_n(t),\omega)\\ &= \int_0^{T_n} X^{L_\alpha}\frac{\partial\phi}{\partial x} + Y^{L_\alpha}\frac{\partial\phi}{\partial v},\end{aligned}$$

dividing by T_n and letting $n\to\infty$. □

5 Graph property, regularity and stationary Mather measures

In this section establish that the discounted Mather measures are supported in a graph of a (partially) Lipschitz function. As we are using similar techniques to [EG01] (see also [BG08]) we will present in this section the main differences and technical points and postpone to Appendix C the detailed proofs. We will the discounted Mather measures to construct a stationary Mather measure invariant under the Euler-Lagrange flow.

We assume that
$$L(x+y,v,\omega) - L(x,v,\omega) \le (c+cL)|y|$$

Lemma 10. Let u_α be the viscosity solution of $H(0,D_xu_\alpha(0,\omega),\omega) + \alpha u_\alpha(0,\omega) = 0$ given by Proposition 4. Then
$$\lim_{\alpha\to 0}\alpha u_\alpha(0,\omega)$$
does not depend on ω.

Proof. We know that αu_α is uniformly bounded, so $\alpha u_\alpha(0,\omega)\to\xi(\omega)$ pointwise for some function. On the other hand, fixed $\omega_0\in\Omega$ we know that $u_\alpha(y,\omega_0)$ is uniformly Lipschitz in x, uniformly as $\alpha\to 0$, that is,
$$|u_\alpha(x_1,\omega_0) - u_\alpha(x_2,\omega_0)| < C|x_1-x_2|.$$

Thus, if $|y| < R$ then,

$$\lim_{\alpha\to 0} |\alpha u_\alpha(y,\omega_0) - \alpha u_\alpha(0,\omega_0)| < \lim_{\alpha\to 0} \alpha C|y| = 0,$$

that is, $\lim_{\alpha\to 0} \alpha u_\alpha(0,\tau_y\omega_0) = \lim_{\alpha\to 0} \alpha u_\alpha(0,\omega_0)$ for $|y| < R$.

From Proposition 4 we know that $u_\alpha(0,\omega)$ is Lipschitz in ω with Lipschitz constant K/α, that is,

$$|u_\alpha(0,\omega_1) - u_\alpha(0,\omega_2)| < \frac{K}{\alpha} d(\omega_1,\omega_2).$$

Consider $\varepsilon > 0$ and $y \in \mathbb{R}^n$, such that $d(\tau_y\omega_0,\omega_1) < \varepsilon$. Observe that,

$$|\alpha u_\alpha(0,\omega_0) - \alpha u_\alpha(0,\omega_1)| \leq$$

$$\leq |\alpha u_\alpha(0,\omega_0) - \alpha u_\alpha(0,\tau_y\omega_0)| + |\alpha u_\alpha(0,\tau_y\omega_0) - \alpha u_\alpha(0,\omega_1)| \leq$$

$$\leq |\alpha u_\alpha(0,\omega_0) - \alpha u_\alpha(0,\tau_y\omega_0)| + \alpha\frac{K}{\alpha} d(\tau_y\omega_0,\omega_1).$$

Sending $\alpha \to 0$, and then $\varepsilon \to 0$ we get $\lim_{\alpha\to 0} \alpha u_\alpha(0,\omega_0) = \lim_{\alpha\to 0} \alpha u_\alpha(0,\omega_1)$. Thus, $\xi(\omega)$ is constant. □

Lemma 11. Let u_α be the viscosity solution of $H(0, D_x u_\alpha(0,\omega),\omega) + \alpha u_\alpha(0,\omega) = 0$ given by Proposition 4. Then

$$\lim_{\alpha\to 0} \alpha u_\alpha(0,\omega) = \overline{H},$$

where

$$\overline{H} = \inf \int_{\mathbb{R}^n\times\Omega} L d\mu,$$

and the infimum is taken over all stationary holonomic measures.

Proof. Denote by $\tilde{H}$ the limit as $\alpha \to 0$ of αu_α, which is constant by the previous lemma. Let μ_α be a holonomic discounted stationary Mather measure. Then, because μ_α is holonomic we have

$$\overline{H} \leq \lim_{\alpha\to 0} \int_{\mathbb{R}^n\times\Omega} L d\mu_\alpha = \lim_{\alpha\to 0} \alpha \int_{\mathbb{R}^n\times\Omega} u_\alpha d\mu_\alpha = \tilde{H}.$$

Let μ be a stationary Mather measure. Then, because μ is a discounted holonomic measure with trace μ we have

$$\overline{H} = \int_{\mathbb{R}^n\times\Omega} L d\mu \geq \alpha \int_{\mathbb{R}^n\times\Omega} u_\alpha d\mu \to \tilde{H},$$

as $\alpha \to 0$. This shows that $\tilde{H} = \overline{H}$. □

Lemma 12. Let μ_α be a sequence of discounted stationary Mather measures with trace ν_α. Suppose that $\mu_\alpha \to \mu$ when $\alpha \to 0$, then μ is a stationary Mather measure.

Proof. First we must to prove that μ is a holonomic probability measure. In fact, for any $\varphi \in C^1_s$,

$$\int_{\mathbb{R}^n\times\Omega} v \cdot D_x\varphi(0,\omega) d\mu_\alpha = \alpha \int_{\mathbb{R}^n\times\Omega} \varphi(0,\omega) d\mu_\alpha - \alpha \int_{\Omega} \varphi(0,\omega) d\nu_\alpha \to 0,$$

when $\alpha \to 0$.

Using Corollary 6 we get

$$\int_{\mathbb{R}^n\times\Omega} L d\mu = \lim_{\alpha\to 0} \int_{\mathbb{R}^n\times\Omega} L d\mu_\alpha = \lim_{\alpha\to 0} \bar{H}_\alpha = \lim_{\alpha\to 0} \int_{\Omega} \alpha u_\alpha(0,\omega) d\nu_\alpha(\omega) = \bar{H}.$$

Thus μ is a Mather measure. □

Theorem 13. Let μ_α be a discounted Mather measure with trace ν_α (or if $\alpha = 0$ a stationary Mather measure). Then μ_α is supported in a graph, that is, there exists a measurable function $V_\alpha : \Omega \to \mathbb{R}^n$ such that,

$$\operatorname{supp} \mu_\alpha = \{(v, \omega) \in \mathbb{R}^n \times \Omega | v = V_\alpha(\omega)\}.$$

Proof. As in [BG08], for instance, we just observe that the result follows from the fact that the Lagrangian is strictly convex in v, whereas the discounted holonomy constraint is linear in v. □

Since the holonomic discounted measures are also holonomic, the same techniques in [EG01] (see also [BG08]) can be adapted to establish the following regularity result:

Theorem 14. Let μ_α be a holonomic discounted Mather measure. If u_α is a viscosity solution of (10), then for each $y \in \mathbb{R}$,

$$|D_x u_\alpha(y, \omega) - D_x u_\alpha(0, \omega)| \leq C|y|,$$

θ almost everywhere and uniformly in α.

The proof of this theorem since it follows (almost) exactly the same steps as in [EG01] (see also [BG08]) and is presented for completeness in appendix C. The only difference is the term αu in the Hamilton-Jacobi equation, which can be controlled, as discussed in section 3, because we are using holonomic discounted measures. As a corollaries to the previous Theorem we have

Corollary 15. Let μ_α be a holonomic discounted Mather measure. Then, there exists a function $V_\alpha : \Omega \to \mathbb{R}^n$, such that $\operatorname{supp} \mu_\alpha = \{(v, \omega) \in \mathbb{R}^n \times \Omega | v = V_\alpha(\omega)\}$. Furthermore, V_α is partially Lipschitz in the following sense:

$$|V_\alpha(\tau_y \omega) - V_\alpha(\omega)| \leq C|y|,$$

for all ω in the support of μ_α, and C is uniformly bounded as $\alpha \to 0$.

Finally, our last result concerns the existence of stationary Mather measures invariant under the Euler-Lagrange flow.

Theorem 16. There exists a stationary Mather measure μ which is invariant under the Euler-Lagrange flow. Furthermore μ is supported on a graph.

Proof. Let μ_α be holonomic discounted Mather measures as constructed in theorem 8. Consider a weak limit μ. By lemma 12, μ is a stationary Mather measure. Because for any $\phi(x, v)$ we have

$$\int_{\mathbb{R}^n \times \Omega} W^{L_0} \nabla_{x,v} \phi(0, v, \omega) d\mu_\alpha = \alpha \int_{\mathbb{R}^n \times \Omega} (D^2_{vv} L)^{-1} D_v L D_v \phi(0, v, \omega) d\mu_\alpha.$$

we conclude that

$$\int_{\mathbb{R}^n \times \Omega} W^{L_0} \nabla_{x,v} \phi(0, v, \omega) d\mu = 0.$$

The graph property of stationary Mather measures follows from theorem 13. □

A Proof of Theorem 1

In this appendix we present the proof of Theorem 1, as well as some background material.

Let γ be as in (4). Let $\mathcal{M}$ be the set of weighted Radon measures on $\Omega \times \mathbb{R}^n$, i.e.,

$$\mathcal{M} = \{\text{signed measures on } \mathbb{R}^n \times \Omega \text{ with } \int_{\mathbb{R}^n \times \Omega} \gamma d|\mu| < \infty\}.$$

Note that $\mathcal{M}$ is the dual of the set $C^0_\gamma(\mathbb{R}^n\times\Omega)$.

We introduce the following sets

$$\mathcal{M}_1=\{\mu\in\mathcal{M}|\ \mu \text{ is a positive probability measure}\},$$

and

$$\mathcal{M}_2=\{\mu\in\mathcal{M}|\int_{\mathbb{R}^n\times\Omega} v\cdot D_x\varphi(0,\omega)d\mu(v,\omega)=0,\ \text{for all } \varphi\in C^1_s(\mathbb{R}^n\times\Omega)\}.$$

Using this notation the Mather problem can be reformulated as

$$\min_{\mathcal{M}_1\cap\mathcal{M}_2}\int_{\mathbb{R}^n\times\Omega} L(0,v,\omega)d\mu(v,\omega).$$

Consider the following subset of functions $\phi:\mathbb{R}^n\times\Omega\to\mathbb{R}$,

$$\mathcal{C}=\mathrm{cl}\{\phi\mid\phi(v,\omega)=v\cdot D_x\varphi(0,\omega),\ \text{for some } \varphi\in C^1_s(\mathbb{R}^n\times\Omega)\}.$$

Observe that $\mathcal{C}$ is a closed convex set.

For $\phi\in C^0_\gamma$ let

$$h(\phi)=\sup_{\mathbb{R}^n\times\Omega}(-\phi(v,\omega)-L(0,v,\omega)). \tag{15}$$

Since h is the supremum of linear functions, it is a convex function on C^0_γ. As we will see bellow in Lemma 18, h is a continuous function.

For $\phi\in C^0_\gamma$, let

$$g(\phi)=\begin{cases}0 & \text{if} \quad \phi\in\mathcal{C}\\ -\infty & \text{otherwise.}\end{cases} \tag{16}$$

As $\mathcal{C}$ is a closed convex set we have that g is a concave and upper semicontinuous function. Therefore its Legendre-Fenchel transform is given by

$$g^*(\mu)=\inf_{\phi\in C^0_\gamma(\mathbb{R}^n\times\Omega)}\left(-\int_{\mathbb{R}^n\times\Omega}\phi d\mu-g(\phi)\right). \tag{17}$$

Since h is a convex and lower semicontinuous function, its Legendre-Fenchel transform is given by

$$h^*(\mu)=\sup_{\phi\in C^0_\gamma(\mathbb{R}^n\times\Omega)}\left(-\int_{\mathbb{R}^n\times\Omega}\phi d\mu-h(\phi)\right). \tag{18}$$

Proposition 17. Let g and h defined as in (15) and (16). Then

$$h^*(\mu)=\begin{cases}\int_{\mathbb{R}^n\times\Omega}L(0,v,\omega)d\mu(v,\omega) & \text{if} \quad \mu\in\mathcal{M}_1\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$g^*(\mu)=\begin{cases}0 & \text{if} \quad \mu\in\mathcal{M}_2\\ -\infty & \text{otherwise.}\end{cases}$$

Proof. First we assume that $\mu\in\mathcal{M}_1$. As h is a convex function, its Legendre transform is given by (18). Using the definition of h, equation (15), we get

$$h^*(\mu)=\sup_{\phi\in C^0_\gamma(\mathbb{R}^n\times\Omega)}\left(-\int_{\mathbb{R}^n\times\Omega}\phi d\mu-\sup_{\mathbb{R}^n\times\Omega}(-\phi(v,\omega)-L(0,v,\omega))\right).$$

Consider the family of compact subsets of $\mathbb{R}^n\times\Omega$ given by

$$K_n=\{(v,\omega)\in\mathbb{R}^n\times\Omega\,|\,|v|\le n\},$$

and let $\eta_n : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a continuous function such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in K_n, and $\operatorname{supp} \eta_n \subset K_{n+1}$. Then define

$$L_n = L(0, v, \omega) \cdot \eta_n(v, \omega).$$

Observe that the sequence L_n is increasing and pointwise convergent to $L(0, v, \omega)$.

Is easy to see that $L_n \in C^0_\gamma(\mathbb{R}^n \times \Omega)$. Furthermore, for fixed n, one can write any function $\phi \in C^0_\gamma(\mathbb{R}^n \times \Omega)$ as $\phi = -L_n + \psi$ where $\psi \in C^0_\gamma(\mathbb{R}^n \times \Omega)$. From this observation we get

$$\begin{aligned}
h^*(\mu) &= \sup_{\psi \in C^0_\gamma(\mathbb{R}^n \times \Omega)} \left(-\int_{\mathbb{R}^n \times \Omega} (-L_n + \psi) d\mu - \sup_{\mathbb{R}^n \times \Omega} (-(-L_n + \psi) - L) \right) \\
&= \sup_{\psi \in C^0_\gamma(\mathbb{R}^n \times \Omega)} \left(\int_{\mathbb{R}^n \times \Omega} L_n d\mu - \int_{\mathbb{R}^n \times \Omega} \psi d\mu - \sup_{\mathbb{R}^n \times \Omega} (L_n - L - \psi) \right) \\
&= \int_{\mathbb{R}^n \times \Omega} L_n d\mu + \sup_{\psi \in C^0_\gamma(\mathbb{R}^n \times \Omega)} \int_{\mathbb{R}^n \times \Omega} \left(-\psi - \sup_{\mathbb{R}^n \times \Omega} (L_n - L - \psi) \right) d\mu. \qquad (19)
\end{aligned}$$

If we take $\psi = 0$ in (19) we have

$$h^*(\mu) \geq \int_{\mathbb{R}^n \times \Omega} L_n d\mu + \int_{\mathbb{R}^n \times \Omega} \left(-\sup_{\mathbb{R}^n \times \Omega} (L_n - L) \right) d\mu \geq \int_{\mathbb{R}^n \times \Omega} L_n d\mu.$$

Thus using the monotone convergence theorem we get

$$h^*(\mu) \geq \int_{\mathbb{R}^n \times \Omega} L d\mu.$$

In order to get the other inequality we can rewrite (19) as follows

$$\begin{aligned}
h^*(\mu) &= \int_{\mathbb{R}^n \times \Omega} L_n d\mu + \sup_{\psi \in C^0_\gamma} \int_{\mathbb{R}^n \times \Omega} \left((S - \psi - S) - \sup_{\mathbb{R}^n \times \Omega} (S - \psi) \right) d\mu \\
&= \int_{\mathbb{R}^n \times \Omega} L_n d\mu + \sup_{\psi \in C^0_\gamma} \int_{\mathbb{R}^n \times \Omega} \left((S - \psi) - \sup_{\mathbb{R}^n \times \Omega} (S - \psi) \right) d\mu - \int_{\mathbb{R}^n \times \Omega} S d\mu,
\end{aligned}$$

where $S = L_n - L$. Since $\mu \in \mathcal{M}_1$, we have $\int_{\mathbb{R}^n \times \Omega} \left((S - \psi) - \sup_{\mathbb{R}^n \times \Omega} (S - \psi) \right) d\mu \leq 0$. Therefore

$$h^*(\mu) \leq \int_{\mathbb{R}^n \times \Omega} L_n d\mu - \int_{\mathbb{R}^n \times \Omega} S d\mu = \int_{\mathbb{R}^n \times \Omega} L d\mu.$$

If $\mu \notin \mathcal{M}_1$, we have two possibilities. First, if $\mu \ngeq 0$ then we can find a positive function $\psi \in C^0_\gamma(\mathbb{R}^n \times \Omega)$ such that $\int \psi d\mu < 0$. Define $\psi_n = n\psi \in C^0_\gamma(\mathbb{R}^n \times \Omega)$, then

$$\begin{aligned}
h^*(\mu) &\geq \left(-\int_{\mathbb{R}^n \times \Omega} \psi_n d\mu - \sup_{\mathbb{R}^n \times \Omega} (-\psi_n - L) \right) \\
&= n \left(\int_{\mathbb{R}^n \times \Omega} -\psi d\mu + \inf_{\mathbb{R}^n \times \Omega} (\psi + \frac{1}{n} L) \right) \to +\infty,
\end{aligned}$$

when $n \to \infty$.

On the other hand, if $\mu \geq 0$ but $\int d\mu \neq 1$ we take $\phi = k \in \mathbb{R}$, then

$$\begin{aligned}
h^*(\mu) &\geq \left(-\int_{\mathbb{R}^n \times \Omega} k d\mu - \sup_{\mathbb{R}^n \times \Omega} (-k - L) \right) \\
&= k \left(1 - \int_{\mathbb{R}^n \times \Omega} d\mu \right) + \inf_{\mathbb{R}^n \times \Omega} L \to +\infty
\end{aligned}$$

when $k \to \pm\infty$, because $L \geq 0$.

Now we compute the Legendre transform of g. As g is concave we compute its Legendre-Fenchel transform using (17). First we suppose $\mu \in \mathcal{M}_2$. In this case we have two possibilities, if $\phi \in \mathcal{C}$ then

$$-\int_{\mathbb{R}^n \times \Omega} \phi d\mu - g(\phi) = 0,$$

else if, $\phi \notin \mathcal{C}$ then

$$-\int_{\mathbb{R}^n \times \Omega} \phi d\mu - g(\phi) = -\int_{\mathbb{R}^n \times \Omega} \phi d\mu - (-\infty) = +\infty$$

thus $g^*(\mu) = 0$.

Otherwise, if $\mu \notin \mathcal{M}_2$ there exists $\phi(v,\omega) = v \cdot D_x\varphi(0,\omega) \in \mathcal{C}$ such that $\int_{\mathbb{R}^n \times \Omega} \phi d\mu > 0$. Define $\phi_\lambda = \lambda v \cdot D_x\varphi(0,\omega) \in \mathcal{C}$ then

$$g^*(\mu) \leq \left(-\int_{\mathbb{R}^n \times \Omega} \phi_\lambda d\mu - g(\phi_\lambda)\right) = -\lambda \int_{\mathbb{R}^n \times \Omega} \phi d\mu \to -\infty$$

when $\lambda \to +\infty$. $\square$

Remark 3. Observe that

$$\min_{\mathcal{M}_1 \cap \mathcal{M}_2} \int_{\mathbb{R}^n \times \Omega} L(0,v,\omega) d\mu(v,\omega) = \min_{\mathcal{M}} (h^*(\mu) - g^*(\mu)).$$

In fact,

$$h^*(\mu) - g^*(\mu) = \begin{cases} \int L(0,v,\omega) d\mu(v,\omega) - 0 & \text{if} \quad \mu \in \mathcal{M}_1 \cap \mathcal{M}_2 \\ \int L(0,v,\omega) d\mu(v,\omega) - (-\infty) & \text{if} \quad \mu \in \mathcal{M}_1 \text{ and } \mu \notin \mathcal{M}_2 \\ +\infty - (0) & \text{if} \quad \mu \notin \mathcal{M}_1 \text{ and } \mu \in \mathcal{M}_2 \\ +\infty - (-\infty) & \text{if} \quad \mu \notin \mathcal{M}_1 \text{ and } \mu \notin \mathcal{M}_2. \end{cases}$$

Lemma 18. The function

$$h(\phi) = \sup_{\mathbb{R}^n \times \Omega} (\phi(v,\omega) - L(0,v,\omega))$$

is continuous.

Proof. Let ϕ_0 be an arbitrary, but fixed, function in $C^0_\gamma(\mathbb{R}^n \times \Omega)$. Suppose $\phi_n \to \phi_0$, that is $\|\phi_n - \phi_0\|_\gamma \to 0$. Let $B_\varepsilon(\phi_0) = \left\{\phi \in C^0_\gamma(\mathbb{R}^n \times \Omega) \mid \|\phi_n - \phi\|_\gamma < \varepsilon\right\}$ be the ball of radius ε centered in ϕ_0. Take $\phi \in B_\varepsilon(\phi_0)$. Since, $\lim_{|v|\to\infty} \frac{|v|}{\gamma(v)} = 0$, $\lim_{|v|\to\infty} \frac{L(0,v,\omega)}{\gamma(v)} = +\infty$ and $\lim_{|v|\to\infty} \frac{|\phi_0(v,\omega)|}{\gamma(v)} \to 0$ uniformly on $\omega \in \Omega$, given $\delta, M > 0$, there exists $R > 0$ such that

$$\begin{cases} \frac{\gamma(v)}{|v|} > \frac{1}{\delta} & \text{if} \quad |v| \quad > R \\ |\frac{\phi_0(v,\omega)}{\gamma(v)}| < \delta & \text{if} \quad |v| \quad > R \\ \frac{L(0,v,\omega)}{\gamma(v)} > M & \text{if} \quad |v| \quad > R. \end{cases}$$

Then, for $|v| > R$,

$$-\phi(v,\omega) - L(0,v,\omega) = \left\{\frac{-\phi(v,\omega) + \phi_0(v,\omega)}{\gamma(v)} + \frac{-\phi_0(v,\omega)}{\gamma(v)} - \frac{L(0,v,\omega)}{\gamma(v)}\right\}\gamma(v) <$$

$$< \frac{\gamma(v)}{|v|}(\varepsilon + \delta - M)|v| \to -\infty$$

when $|v| \to +\infty$.

As
$$\|\phi_n - \phi_0\|_\gamma = \sup_{\mathbb{R}^n \times \Omega} \frac{|(\phi_n - \phi_0)(v,\omega)|}{\gamma(v)} \to 0$$
we have that, for n big enough, we can choose R in such way that
$$h(\phi) = \sup_{\{|v| \leq R\} \times \Omega} (-\phi(v,\omega) - L(0,v,\omega)),$$
and
$$h(\phi_n) = \sup_{\{|v| \leq R\} \times \Omega} (-\phi_n(v,\omega) - L(0,v,\omega)).$$

Since the convergence $-\phi_n - L(0,v,\omega) \to -\phi_0 - L(0,v,\omega)$ is uniform on the compact $\{|v| \leq R\} \times \Omega$, we have
$$\lim_{n\to\infty} h(\phi_n) = \lim_{n\to\infty} \sup_{\{|v| \leq R\} \times \Omega} (-\phi_n(v,\omega) - L(0,v,\omega)) =$$
$$= \sup_{\{|v| \leq R\} \times \Omega} \lim_{n\to\infty} (-\phi_n(v,\omega) - L(0,v,\omega)) = h(\phi_0).$$
Thus the lemma is proved. □

The last ingredient of the duality is the Legendre-Fenchel-Rockafellar Theorem, see for instance [Vil03].

Theorem 19. (Legendre-Fenchel-Rockafellar) Let E be a locally convex Hausdorff topological vectorial space over $\mathbb{R}$ with dual E^*. Suppose that $h : E \to (-\infty, +\infty]$ is convex and lower semicontinuous and $g : E^* \to [-\infty, +\infty)$ is concave and upper semicontinuous. Then
$$\min_{E^*}(h^* - g^*) = \sup_E (g - h),$$
provided that h or g is continuous at some point where both functions are finite. It is part of the theorem that the left hand side is a minimum.

Lemma 20. Define the functional, $S(\phi) = g(\phi) - h(\phi)$. Then S is uniformly continuous in the interior of $\mathcal{C}$.

Proof. In fact, given $\varepsilon > 0$, if $\|\phi_1 - \phi_2\|_\gamma < \varepsilon$, that is, $-\varepsilon\gamma(v) < \phi_1(v,w) - \phi_2(v,w) < \varepsilon\gamma(v)$, for all (v,w), then
$$|S(\phi_1) - S(\phi_2)| < \{\inf \gamma(v)\}\, \varepsilon.$$
In particular
$$\sup_{\phi \in \mathcal{C}} g(\phi) - h(\phi) = \sup_{\substack{\phi = v \cdot D_x \varphi(0,\omega) \\ \varphi \in C^1_s}} g(\phi) - h(\phi).$$
□

B Proof of Proposition 4

Proof of Proposition 4. We must to prove that the function u_α is stationary. Since $L \geq 0$, u_α is well defined as an infimum. On the other hand the stationarity is an easy consequence of the correspondence between the set of all globally Lipschitz trajectories with initial condition

$x(0) = x$ and the set of all globally Lipschitz trajectories with initial condition $y(0) = 0$, given by, $\{x(t)\} \to \{y(t) = x(t) - x\}$. In fact,

$$
\begin{aligned}
u_\alpha(0, \tau_x \omega) &= \inf_{y(0)=0} \int_0^{+\infty} e^{-\alpha t} L(y(t), \dot{y}(t), \tau_x \omega) dt \\
&= \inf_{x(0)=x} \int_0^{+\infty} e^{-\alpha t} L((x(t) - x) + x, \dot{x}(t), \omega) dt = u_\alpha(x, \omega).
\end{aligned}
$$

In order to prove that u_α is a viscosity solution in ω, let $\varphi : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a stationary function such that $u_\alpha(0, \omega) - \varphi(0, \omega)$ has a local minimum (resp. maximum) in $\omega_\varphi \in \Omega$ and $u_\alpha(0, \omega_\varphi) - \varphi(0, \omega_\varphi) = 0$.

Consider a trajectory satisfying $x(0) = 0$ such that $x(t)$ is a finite time minimizing, globally Lipschitz trajectory, for the dynamic programing principle (12), that is,

$$
u_\alpha(0, \omega_\varphi) = \int_0^T e^{-\alpha t} L(x(t), \dot{x}(t), \omega_\varphi) dt + e^{-\alpha T} u_\alpha(x(T), \omega_\varphi), \tag{20}
$$

for T small enough.

Suppose that $\mathcal{H}(\varphi, \omega_\varphi) < 0$, by continuity there is a neighborhood B of ω_φ in Ω and $\delta > 0$ such that $\mathcal{H}(\varphi, \omega) < -\delta$ for all $\omega \in B$. Since $\mathcal{H}(\varphi, \omega) = H(0, D_x \varphi(0, \omega), \omega) + \alpha \varphi(0, \omega)$ we have $-v D_x \varphi(0, \omega) - L(0, v, \omega) + \alpha \varphi(0, \omega) < -\delta$, for all $\omega \in B$ and $v \in \mathbb{R}^n$. If we choose $v = \dot{x}(t)$ and $\omega = \tau_{x(t)} \omega_\varphi$ then

$$
\dot{x}(t) D_x \varphi(0, \tau_{x(t)} \omega_\varphi) + L(x(t), \dot{x}(t), \omega_\varphi) - \alpha \varphi(x(t), \omega_\varphi) > \delta,
$$

for $0 < t < T$.

Integrating this expression and using $\frac{d}{dt} \varphi(x(t), \omega) = \dot{x}(t) D_x \varphi(0, \tau_{x(t)} \omega)$ we get,

$$
\varphi(0, \tau_{x(T)} \omega_\varphi) - \varphi(0, \omega_\varphi) + \int_0^T L(x(t), \dot{x}(t), \omega_\varphi) dt - \alpha \int_0^T \varphi(x(t), \omega_\varphi) dt > \delta T.
$$

Since $u_\alpha(0, \omega) \geq \varphi(0, \omega)$ in B and $u_\alpha(0, \omega_\varphi) = \varphi(0, \omega_\varphi)$, we have

$$
u_\alpha(0, \tau_{x(T)} \omega_\varphi) - u_\alpha(0, \omega_\varphi) + \int_0^T L(x(t), \dot{x}(t), \omega_\varphi) dt - \alpha \int_0^T \varphi(x(t), \omega_\varphi) dt > \delta T.
$$

Using (20) in the last inequality we get,

$$
(1 - e^{-\alpha T}) u_\alpha(0, \tau_{x(T)} \omega_\varphi) + \int_0^T (1 - e^{-\alpha t}) L(x(t), \dot{x}(t), \omega_\varphi) dt - \alpha \int_0^T \varphi(x(t), \omega_\varphi) dt > \delta T.
$$

Writing

$$
u_\alpha(0, \tau_{x(T)} \omega_\varphi) + \frac{T}{(1 - e^{-\alpha T})} \frac{1}{T} \int_0^T (1 - e^{-\alpha t}) L(x(t), \dot{x}(t), \omega_\varphi) dt -
$$

$$
\alpha \frac{T}{(1 - e^{-\alpha T})} \frac{1}{T} \int_0^T \varphi(x(t), \omega_\varphi) dt > \delta \frac{T}{1 - e^{-\alpha T}}
$$

and using $\lim_{T \to 0} \dfrac{T}{1 - e^{-\alpha T}} = \dfrac{1}{\alpha}$, we get

$$
u_\alpha(0, \omega_\varphi) - \varphi(0, \omega_\varphi) > \frac{\delta}{\alpha}
$$

contradicting $u_\alpha(0, \omega_\varphi) = \varphi(0, \omega_\varphi)$.

The proof for the maximum case is analogous and so the theorem is proved. $\square$

C Proof of Theorem 14

In this last appendix we give a proof of Theorem 14. Before that we need to establish some additional results. We note here that we will be using the techniques in [EG01] (see also [BG08]) adapted to the stationary setting.

Remark 4. Let u_α be a viscosity solution in ω of (10), then, because it is also a viscosity solution in x (Proposition 2) and it is Lipschitz, $D_x u_\alpha(0, \tau_y \omega)$ is defined Lebesgue almost everywhere and

$$H(y, D_x u_\alpha(y,\omega), \omega) + \alpha u_\alpha(y,\omega) = 0,$$

for Lebesgue almost everywhere $y \in \mathbb{R}^n$.

For any probability measure μ, we can define a new measure of probability $\tilde{\mu}$ in $\mathbb{R}^n \times \Omega$ given by,

$$\int_{\mathbb{R}^n\times\Omega} \psi(p,\omega) d\tilde{\mu}(p,\omega) = \int_{\mathbb{R}^n\times\Omega} \psi(-D_v L(0,v,\omega),\omega) d\mu(v,\omega).$$

In this case, the integral holonomy constraint can be rewritten as

$$\int_{\mathbb{R}^n\times\Omega} D_p H(0,p,\omega)\cdot D_x\varphi(0,\omega) d\tilde{\mu}(p,\omega) = 0,$$

$\forall \varphi \in C^1_s(\mathbb{R}^n \times \Omega)$.

Theorem 21. Let μ_α be a holonomic discounted stationary Mather measure. Denote the projection in the coordinate ω of μ_α by θ_α, that is

$$\int_\Omega \varphi(\omega) d\theta_\alpha = \int_{\mathbb{R}^n\times\Omega} \varphi(\omega) d\mu_\alpha.$$

If u_α is a viscosity solution of (10), then $D_x u_\alpha(0,\omega)$ exists θ_α-a.e, and $\tilde{\mu}_\alpha$-a.e, $p = -D_x u_\alpha(0,\omega)$.

Proof. By the strict uniform continuity of H there exists $\gamma > 0$ such that for any $p, q, y \in \mathbb{R}^n$ and $\omega \in \Omega$ we have

$$H(0,p,\tau_y\omega) \geq H(0,q,\tau_y\omega) + D_p H(0,q,\tau_y\omega)(p-q) + \frac{\gamma}{2}|p-q|^2.$$

Let $u^\varepsilon = u * \eta$, by Remark 4, for almost every ω and y, let $p = D_x u_\alpha(0,\tau_y\omega)$ and $q = D_x u^\varepsilon_\alpha(0,\omega)$. Then

$$\begin{aligned} &H(0, D_x u_\alpha(0,\tau_y\omega), \tau_y\omega) \geq H(0, D_x u^\varepsilon_\alpha(0,\omega), \tau_y\omega) \\ &\quad + D_p H(0, D_x u^\varepsilon_\alpha(0,\omega), \tau_y\omega)(D_x u_\alpha(0,\tau_y\omega) - D_x u^\varepsilon_\alpha(0,\omega)) + \frac{\gamma}{2}|D_x u_\alpha(0,\tau_y\omega) - D_x u^\varepsilon_\alpha(0,\omega)|^2. \end{aligned}$$

Multiplying by $\eta^\varepsilon(y)$ and integrating we get

$$\begin{aligned} &\int_{\mathbb{R}^n} H(0, D_x u^\varepsilon_\alpha(0,\omega), \tau_y\omega)\eta^\varepsilon(y)dy + \int_{\mathbb{R}^n} \frac{\gamma}{2}|D_x u_\alpha(0,\tau_y\omega) - D_x u^\varepsilon_\alpha(0,\omega)|^2\eta^\varepsilon(y)dy \\ &\leq \int_{\mathbb{R}^n} H(0, D_x u_\alpha(0,\tau_y\omega), \tau_y\omega)\eta^\varepsilon(y)dy \\ &+ \int_{\mathbb{R}^n} D_p H(0, D_x u^\varepsilon_\alpha(0,\omega), \tau_y\omega)\left[D_x u^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\tau_y\omega)\right]\eta^\varepsilon(y)dy. \end{aligned}$$

Remark 4 implies that, $H(y, D_x u_\alpha(y,\omega),\omega) = -\alpha u_\alpha(y,\omega)$ almost everywhere y. Thus

$$\int_{\mathbb{R}^n} H(0, D_x u^\varepsilon_\alpha(0,\tau_y\omega), \tau_y\omega)\eta^\varepsilon(y)dy + \beta^\varepsilon(\omega) \leq -\alpha u^\varepsilon_\alpha(0,\omega) + o_\omega(\varepsilon) \tag{21}$$

where

$$\beta^\varepsilon(\omega) = \int_{\mathbb{R}^n} \frac{\gamma}{4}|D_x u_\alpha(0,\tau_y\omega) - D_x u_\alpha^\varepsilon(0,\tau_y\omega)|^2 \eta^\varepsilon(y)dy.$$

On the other hand, the convexity of H, implies that,

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\Omega} \frac{\gamma}{2}|D_x u_\alpha^\varepsilon(0,\omega) - p|^2 d\tilde{\mu}_\alpha(p,\omega)\\
&\leq \int_{\mathbb{R}^n\times\Omega} \left[H(0, D_x u_\alpha^\varepsilon(0,\omega),\omega) - H(0,p,\omega) - D_pH(0,p,\omega)(D_x u_\alpha^\varepsilon(0,\omega) - p)\right] d\tilde{\mu}_\alpha(p,\omega)\\
&= \int_{\mathbb{R}^n\times\Omega} H(0, D_x u_\alpha^\varepsilon(0,\omega),\omega) d\tilde{\mu}_\alpha(p,\omega)\\
&- \int_{\mathbb{R}^n\times\Omega} \left[H(0,p,\omega) + D_pH(0,p,\omega)D_x u_\alpha^\varepsilon(0,\omega) - D_pH(0,p,\omega)p\right] d\tilde{\mu}_\alpha(p,\omega)\\
&= \int_{\mathbb{R}^n\times\Omega} H(0, D_x u_\alpha^\varepsilon(0,\omega),\omega) + L(0, -D_pH(0,p,\omega),\omega) d\tilde{\mu}_\alpha(p,\omega)\\
&= \int_{\mathbb{R}^n\times\Omega} H(0, D_x u_\alpha^\varepsilon(0,\omega),\omega) d\tilde{\mu}_\alpha(p,\omega) + \bar{H}_\alpha. \qquad (22)
\end{aligned}$$

Integrating (21) with respect to $\tilde{\mu}$ and adding (22), we get

$$\int_{\mathbb{R}^n\times\Omega} \frac{\gamma}{2}|D_x u_\alpha^\varepsilon(0,\omega) - p|^2 d\tilde{\mu}_\alpha(p,\omega) + \int_\Omega \beta^\varepsilon(\omega) d\theta(\omega) < o(\varepsilon).$$

So, θ_α almost everywhere we have $D_x u_\alpha(0,\omega) = \lim_{\varepsilon\to 0} D_x u_\alpha^\varepsilon(0,\omega)$, in particular $p = D_x u_\alpha(0,\omega)$ in the support of $\tilde{\mu}$. □

Theorem 22. Let μ_α be a holonomic Mather measure for the discounted stationary Mather problem. If u_α is a viscosity solution of (10), then for each $h \in \mathbb{R}$,

$$|u_\alpha(h,\omega) - 2u_\alpha(0,\omega) + u_\alpha(-h,\omega)| \leq C|h|^2,$$

θ almost everywhere.

Proof. If $h \neq 0$ then we define,

$$\tilde{u}_\alpha(x,\omega) = u_\alpha(x+h,\omega) \text{ and } \hat{u}_\alpha(x,\omega) = u_\alpha(x-h,\omega),$$

and $\tilde{u}_\alpha^\varepsilon(x,\omega)$ and $\hat{u}_\alpha^\varepsilon(x,\omega)$, the corresponding smoothings (see Remark 1).

Remember that

$$H(h, D_x\tilde{u}_\alpha^\varepsilon(0,\omega),\omega) + \alpha\tilde{u}_\alpha^\varepsilon(0,\omega) \leq c\varepsilon,$$

and

$$H(-h, D_x\hat{u}_\alpha^\varepsilon(0,\omega),\omega) + \alpha\hat{u}_\alpha^\varepsilon(0,\omega) \leq c\varepsilon.$$

Thus,

$$\begin{aligned}
H(0,&D_x\tilde{u}_\alpha^\varepsilon(0,\omega),\omega) - 2H(0, D_x u_\alpha(0,\omega),\omega) + H(0, D_x\hat{u}_\alpha^\varepsilon(0,\omega),\omega)\\
&= H(0, D_x\tilde{u}_\alpha^\varepsilon(0,\omega),\omega) - H(h, D_x\tilde{u}_\alpha^\varepsilon(0,\omega),\omega) + H(h, D_x\tilde{u}_\alpha^\varepsilon(0,\omega),\omega) + \alpha\tilde{u}_\alpha^\varepsilon(0,\omega)\\
&\quad - \alpha\tilde{u}_\alpha^\varepsilon(0,\omega) + 2\alpha u_\alpha(0,\omega) - \alpha\hat{u}_\alpha^\varepsilon(0,\omega)\\
&\quad + \alpha\hat{u}_\alpha^\varepsilon(0,\omega) + H(-h, D_x\hat{u}_\alpha^\varepsilon(0,\omega),\omega) - H(-h, D_x\hat{u}_\alpha^\varepsilon(0,\omega),\omega) + H(0, D_x\hat{u}_\alpha^\varepsilon(0,\omega),\omega)\\
&\leq 2c\varepsilon - \alpha\left(\tilde{u}_\alpha^\varepsilon(0,\omega) - 2u_\alpha(0,\omega) + \hat{u}_\alpha^\varepsilon(0,\omega)\right)\\
&\quad - \left(D_xH(0, D_x\tilde{u}_\alpha^\varepsilon(0,\omega),\omega) - D_xH(0, D_x\hat{u}_\alpha^\varepsilon(0,\omega),\omega)\right)h + O(|h|^2). \qquad (23)
\end{aligned}$$

On the other hand the convexity of H implies that

$$\begin{aligned}H(0, D_x\tilde{u}^\varepsilon_\alpha(0,\omega),\omega) \geq & H(0, D_x u_\alpha(0,\omega),\omega)\\ & + D_pH(0, D_x u_\alpha(0,\omega),\omega)(D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega))\\ & + \frac{\gamma}{2}|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2,\end{aligned}$$

and

$$\begin{aligned}H(0, D_x\hat{u}^\varepsilon_\alpha(0,\omega),\omega) \geq & H(0, D_x u_\alpha(0,\omega),\omega)\\ & + D_pH(0, D_x u_\alpha(0,\omega),\omega)(D_x\hat{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega))\\ & + \frac{\gamma}{2}|D_x\hat{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2.\end{aligned}$$

Adding these two formulas we obtain the following inequality:

$$\begin{aligned}&\frac{\gamma}{2}\left(|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2 + |D_x\hat{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2\right)\\ &+ D_pH(0, D_x u_\alpha(0,\omega),\omega)(D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - 2D_x u_\alpha(0,\omega) + D_x\hat{u}^\varepsilon_\alpha(0,\omega))\\ &\leq \left(H(0, D_x\tilde{u}^\varepsilon_\alpha(0,\omega),\omega) - 2H(0, D_x u_\alpha(0,\omega),\omega) + H(0, D_x\hat{u}^\varepsilon_\alpha(0,\omega),\omega)\right).\end{aligned}$$

By (23) we have,

$$\begin{aligned}&\frac{\gamma}{2}\left(|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2 + |D_x\hat{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2\right)\\ &+ D_pH(0, D_x u_\alpha(0,\omega),\omega)(D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - 2D_x u_\alpha(0,\omega) + D_x\hat{u}^\varepsilon_\alpha(0,\omega))\\ &\leq 2c\varepsilon - \alpha\left(\tilde{u}^\varepsilon_\alpha(0,\omega) - 2u_\alpha(0,\omega) + \hat{u}^\varepsilon_\alpha(0,\omega)\right)\\ &- \left(D_xH(0, D_x\tilde{u}^\varepsilon_\alpha(0,\omega),\omega) - D_xH(0, D_x\hat{u}^\varepsilon_\alpha(0,\omega),\omega)\right)h + O(|h|^2).\end{aligned}$$

Or equivalently,

$$\begin{aligned}&\frac{\gamma}{2}\left(|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2 + |D_x\hat{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2\right)\\ &+ D_pH(0, D_x u_\alpha(0,\omega),\omega)(D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - 2D_x u_\alpha(0,\omega) + D_x\hat{u}^\varepsilon_\alpha(0,\omega))\\ &+ \alpha\left(\tilde{u}^\varepsilon_\alpha(0,\omega) - 2u_\alpha(0,\omega) + \hat{u}^\varepsilon_\alpha(0,\omega)\right)\\ &\leq 2c\varepsilon - \left(D_xH(0, D_x\tilde{u}^\varepsilon_\alpha(0,\omega),\omega) - D_xH(0, D_x\hat{u}^\varepsilon_\alpha(0,\omega),\omega)\right)(h) + O(|h|^2).\end{aligned} \tag{24}$$

Define, $\beta^\varepsilon(x,\omega) = \tilde{u}^\varepsilon_\alpha(x,\omega) - 2u_\alpha(x,\omega) + \hat{u}^\varepsilon_\alpha(x,\omega)$, so (24) can be rewritten as

$$\begin{aligned}&\frac{\gamma}{2}\left(|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2 + |D_x\hat{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2\right)\\ &+ D_pH(0, D_x u_\alpha(0,\omega),\omega)D_x\beta^\varepsilon(0,\omega) + \alpha\beta^\varepsilon(0,\omega)\\ &\leq 2c\varepsilon - \left(D_xH(0, D_x\tilde{u}^\varepsilon_\alpha(0,\omega),\omega) - D_xH(0, D_x\hat{u}^\varepsilon_\alpha(0,\omega),\omega)\right)(h) + O(|h|^2)\end{aligned} \tag{25}$$

Applying the inequality

$$\begin{aligned}&\|\left(D_xH(0, D_x\tilde{u}^\varepsilon_\alpha(0,\omega),\omega) - D_xH(0, D_x\hat{u}^\varepsilon_\alpha(0,\omega),\omega)\right)h\|\\ \leq&\|D^2pxH(0, D_x\tilde{u}^\varepsilon_\alpha(0,\omega),\omega)\| \cdot |D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x\hat{u}^\varepsilon_\alpha(0,\omega)| \cdot |h|\\ \leq&\frac{\gamma}{4}|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x\hat{u}^\varepsilon_\alpha(0,\omega)|^2 + \frac{1}{\gamma}|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x\hat{u}^\varepsilon_\alpha(0,\omega)|^2 \cdot |h|^2\\ \leq&\frac{\gamma}{4}\left(|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2 + |D_x\hat{u}^\varepsilon_\alpha(0,\omega) - D_x u_\alpha(0,\omega)|^2\right)\\ &+ \frac{1}{\gamma}|D_x\tilde{u}^\varepsilon_\alpha(0,\omega) - D_x\hat{u}^\varepsilon_\alpha(0,\omega)|^2 \cdot |h|^2,\end{aligned}$$

to (25) we get,

$$\frac{\gamma}{4}\left(|D_x\tilde{u}^\varepsilon_\alpha(0,\omega)-D_xu_\alpha(0,\omega)|^2+|D_x\hat{u}^\varepsilon_\alpha(0,\omega)-D_xu_\alpha(0,\omega)|^2\right)$$
$$+D_pH(0,D_xu_\alpha(0,\omega),\omega)D_x\beta^\varepsilon(0,\omega)+\alpha\beta^\varepsilon(0,\omega)\leq 2C(\varepsilon+|h|^2). \tag{26}$$

Consider a function $\Psi:\mathbb{R}\to\mathbb{R}$, such that $\Phi(s)=\Psi'(s)\geq 0$. We can multiply (26) by $\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)$ and integrate with respect to θ,

$$\int_\Omega\frac{\gamma}{4}\left(|D_x\tilde{u}^\varepsilon_\alpha(0,\omega)-D_xu_\alpha(0,\omega)|^2+|D_x\hat{u}^\varepsilon_\alpha(0,\omega)-D_xu_\alpha(0,\omega)|^2\right)\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta$$
$$+\int_\Omega D_pH(0,D_xu_\alpha(0,\omega),\omega)D_x\beta^\varepsilon(0,\omega)\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta$$
$$+\int_\Omega\alpha\beta^\varepsilon(0,\omega)\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta\leq 2C(\varepsilon+|h|^2)\int_\Omega\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta. \tag{27}$$

We have

$$|h|^2\int_\Omega D_pH(0,D_xu_\alpha(0,\omega),\omega)\frac{1}{|h|^2}D_x\beta^\varepsilon(0,\omega)\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta$$
$$=\int_\Omega D_pH(0,D_xu_\alpha(0,\omega),\omega)D_x\Psi\left(\frac{\beta^\varepsilon}{|h|^2}\right)(0,\omega)d\theta=0.$$

Thus, (27) can be restated as,

$$\int_\Omega|D_x\tilde{u}^\varepsilon_\alpha(0,\omega)-D_x\hat{u}^\varepsilon_\alpha(0,\omega)|^2\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta$$
$$+\int_\Omega\alpha\beta^\varepsilon(0,\omega)\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta\leq 2C(\varepsilon+|h|^2)\int_\Omega\Phi\left(\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\right)d\theta. \tag{28}$$

Define, $A_\lambda=\{\omega|\frac{\beta^\varepsilon(0,\omega)}{|h|^2}\leq-\lambda\}$, and consider the function Ψ defined by

$$\Psi(s)=\begin{cases}s & \text{if } s\leq-\lambda,\\ 1 & \text{otherwise.}\end{cases}$$

Fix a positive constant γ such that the functions $\bar{u}_\alpha(x,\omega)=\tilde{u}_\alpha(x,\omega)-\frac{\gamma}{2}|x|^2$ and $\bar{u}^\varepsilon_\alpha(x,\omega)=\tilde{u}^\varepsilon_\alpha(x,\omega)-\frac{\gamma}{2}|x|^2$ are concave. Observe that a point ω is in A_λ only if

$$\bar{u}^\varepsilon_\alpha(h,\omega)-2\bar{u}_\alpha(0,\omega)+\bar{u}^\varepsilon_\alpha(-h,\omega)\leq-(\lambda+\gamma)|h|^2.$$

Define $F^\varepsilon(t)=\bar{u}^\varepsilon_\alpha(t\frac{h}{|h|},\omega)$. Since F^ε is concave and $(F^\varepsilon)''\leq 0$ we have

$$\bar{u}^\varepsilon_\alpha(h,\omega)-2\bar{u}^\varepsilon_\alpha(0,\omega)+\bar{u}^\varepsilon_\alpha(-h,\omega)\geq(D_x\bar{u}^\varepsilon_\alpha(h,\omega)-D_x\bar{u}^\varepsilon_\alpha(-h,\omega))h.$$

Subtracting this inequalities we get,

$$(\lambda+\gamma)|h|^2\leq 2|\bar{u}^\varepsilon_\alpha(0,\omega)-\bar{u}_\alpha(0,\omega)|+|D_x\bar{u}^\varepsilon_\alpha(h,\omega)-D_x\bar{u}^\varepsilon_\alpha(-h,\omega)|\,|h|.$$

Since u_α is stationary and uniformly Lipschitz continuous we have $|\bar{u}^\varepsilon_\alpha(0,\omega)-\bar{u}_\alpha(0,\omega)|\leq C\varepsilon$. thus we can choose ε in such way that

$$|D_x\bar{u}^\varepsilon_\alpha(h,\omega)-D_x\bar{u}^\varepsilon_\alpha(-h,\omega)|\geq(\frac{\lambda}{2}+\gamma)|h|$$

and

$$|D_x u_\alpha^\varepsilon(h,\omega) - D_x u_\alpha^\varepsilon(-h,\omega)| \geq (\frac{\lambda}{2}+\gamma)|h|.$$

Using this estimates in (6) we get

$$(\frac{\lambda}{2}+\gamma)^2|h|^2\theta(A_\lambda) - \alpha\lambda|h|^2\theta(A_\lambda) \leq 2C(\varepsilon+|h|^2)\theta(A_\lambda).$$

Observe that, if $\theta(A_\lambda) > 0$ then the left hand side of this inequality converges to $+\infty$ when $\lambda \to +\infty$, so there exists a value λ_0 such that $\theta(A_\lambda) = 0$, that is, $-\lambda_0|h|^2 \leq \tilde{u}_\alpha^\varepsilon(x,\omega) - 2u_\alpha(x,\omega) + \hat{u}_\alpha^\varepsilon(x,\omega)$, θ almost everywhere. The upper bound comes from the semiconcavity of u_α. Thus there exists $C > 0$ such that $|u_\alpha(h,\omega) - 2u_\alpha(0,\omega) + u_\alpha(-h,\omega)| \leq C|h|^2$, θ almost everywhere, which completes the proof of the theorem. □

Proof of Theorem 14. Let θ be the projection of μ_α. By Theorem 21, $D_x u_\alpha(0,\omega)$ exists θ-a.e. On the other hand, fixed $\omega \in \operatorname{supp}\theta$, $D_x u_\alpha(y,\omega)$ exists Lebesgue almost everywhere.

We claim that

$$|u_\alpha(y,\omega) - u_\alpha(0,\omega) + u_\alpha(-y,\omega)| \leq C|y|^2.$$

This claim is a consequence of Theorem 22, by choosing $h = y$ and of the semi-concavity of u_α. In fact, we have

$$-C|h|^2 \leq u_\alpha(y,\omega) - 2u_\alpha(0,\omega) + u_\alpha(-y,\omega) \leq C|h|^2, \tag{29}$$

$$u_\alpha(y,\omega) - u_\alpha(0,\omega) - D_x u_\alpha(0,\omega)y \leq C|y|^2, \tag{30}$$

and

$$u_\alpha(-y,\omega) - u_\alpha(0,\omega) + D_x u_\alpha(0,\omega)y \leq C|y|^2. \tag{31}$$

The claim is obtained from (30) and from the difference between (29) and (31).

Let $z \in \mathbb{R}$ be a point such that $|z| \leq 2|y|$. The semi-concavity of u_α implies that,

$$u_\alpha(z,\omega) \leq u_\alpha(y,\omega) + D_x u_\alpha(y,\omega)(z-y) + C|z-y|^2. \tag{32}$$

Using, $u_\alpha(z,\omega) = u_\alpha(0,\omega) + D_x u_\alpha(0,\omega)z + o(|z|^2)$ and $u_\alpha(y,\omega) = u_\alpha(0,\omega) + D_x u_\alpha(0,\omega)y + o(|y|^2)$ in (4) we get

$$(D_x u_\alpha(0,\omega) - D_x u_\alpha(y,\omega))(z-y) \leq C|y|^2. \tag{33}$$

If we take $z = y + |y|\frac{D_x u_\alpha(0,\omega) - D_x u_\alpha(y,\omega)}{|D_x u_\alpha(0,\omega) - D_x u_\alpha(y,\omega)|}$ then we obtain $|D_x u_\alpha(y,\omega) - D_x u_\alpha(0,\omega)| \leq C|y|$. □

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